The unified geometric theory of mesoscopic stochastic pumps and reversible ratchets

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(Dated: February 5, 2008)

We construct a unifying theory of geometric effects in mesoscopic stochastic kinetics. We demonstrate that the adiabatic pump and the reversible ratchet effects, as well as similar novel phenomena in other domains, such as in epidemiology, all follow from geometric phase contributions to the effective action in the stochastic path integral representation of the moment generating function of particle fluxes. The theory provides a universal technique for identification, prediction and calculation of pump-like phenomena in an arbitrary mesoscopic stochastic framework.

PACS numbers: 03.65.Vf,05.10.Gg,05.40.Ca

Introduction. A number of effects in classical nonstationary statistical physics, such as the reversible ratchet [1, 2, 3] and the adiabatic pump effects [4, 5], are known (or anticipated) to have a geometric origin. Examples and applications can be found in a variety of fields, including electronic metrology [6], cell motility [7], ion pumping through a cell membrane [8], and dissipative chemical kinetics [9]. The distinct feature of these effects is that, under a slow periodic perturbation, transport coefficients are not a simple average of those in the strict static approximation, but contain an extra component, which changes its sign under a time-reversal of the perturbation. Although these effects have been well-studied in a variety of fields (see [10] for reviews), a general theory, which clearly disambiguates the pump (or ratchet) currents from other nonequilibrium transport, provides a unified view of disparate pump-like phenomena, and suggests universal quantitative methods for calculation of moments of pump fluxes, is still missing.

We address this problem using the recently introduced stochastic path integral representation of the moment generating function (mgf) of fluxes in mesoscopic stochastic systems [12, 13]. In this letter, we demonstrate that the stochastic path integral technique can be employed to calculate moments of pump fluxes in a general stochastic driven system in the mesoscopic (many particles) and the adiabatic (slowly varying external parameters) regimes, and that it makes a clear distinction between the pump fluxes and other currents by relating the former to a geometric phase contribution to the flux mgf. Our theory clarifies the connection between the reversible ratchet and the adiabatic pump and allows to identify similar effects in new contexts, which we demonstrate in a specific scenario from epidemiology.

Pump current from particle exclusion. Let two absorbing states S and P (substrate and product in a Michaelis-Menten enzymatic reaction [5, 11], distinct cellular compartments, neighborhoods of a city), exchange particles (molecules, humans) via an intermediate system B (bin, enzyme, membrane channel, transportation hub). Our goal is to find the $S \to P$ flux J and its fluctuations on time scales much larger than the fluctuation time in the bin B, assuming the mesoscopic regime with a large typical number of particles in the bin, $N \gg 1$.

Particles interact, and the in- and out-going transition rates can depend on the number of particles in the bin, $N(t)$. The simplest example of this kind is when the bin has a finite size, so that $N \leq N_B = \text{const} < \infty$. Then the in-rates are proportional to the number of empty spaces in the bin, while the per-particle out-rates are not affected by the occupancy. The full kinetic scheme is

1. $S \to B$; rate $k_1(N,t) = q_1(t)(N_B - N)$,
2. $B \to S$; rate $k_{-1}(N) = q_{-1}N$,
3. $P \to B$; rate $k_{-2}(N,t) = q_{-2}(t)(N_B - N)$,
4. $B \to P$; rate $k_2(N) = q_2 N$.

We allow q_1 and q_{-2} to undergo a slow periodic modulation with a frequency ω, which can be achieved in the biochemical context by coupling S and P to particle baths with modulated chemical potentials. In other transport problems, such as transportation systems, the same modulation may be produced by time-of-day variations. We note that, unlike in [12], our formulation has three time scales: fast instantaneous jumps among states, equilibration in the bin, and adiabatic changes of the rates.

Now the path integral technique of [12] can be applied. Since $N \gg 1$, there exists a time scale δt, over which many transitions into and out of B happen, but the fractional change in the bin occupancy remains small, $1 \ll \delta N \ll N$. Then the rate changes δk_i, $i = -2, -1, 1, 2$ are also small, and all transitions are uncorrelated and Poissonian. Thus the probability of the number of particle transitions for the i'th reaction over time δt, denoted by δQ_i, can be written as $P(\delta Q_i, t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\chi_i e^{-i\chi_i \delta Q_i + N_B H_i(\chi_i,t)\delta t}$, where $N_B H_i \delta t = k_i(N,t)[\exp(i\chi_i) - 1]\delta t \equiv k_i(N,t) e_{\chi_i} \delta t$ is the mgf of a Poisson process with the mean $k_i \delta t$. Note that we define $e_x \equiv e^{ix} - 1$ for any x.

Our goal is to find the mgf of the net particle number Q_P transfered into P during a long time interval $(0, T)$. This is formally given by an integral over fluxes at each

moment of (discretized) time weighted by probabilities $P(\delta Q_i, t)$ and constrained by particle conservation laws:

$$\langle e^{i\chi_C Q_P}\rangle = \int \prod_{k=1}^{T/\delta t} dN(t_k) \prod_{i=\pm2,\pm1} d\delta Q_i(t_k) P[\delta Q_i(t_k)] \times \\ \times e^{i\chi_C(\delta Q_2(t_k)-\delta Q_{-2}(t_k))}\, \delta[N(t_{k+1}) - N(t_k) \\ - \delta Q_1(t_k) - \delta Q_{-2}(t_k) + \delta Q_{-1}(t_k) + \delta Q_2(t_k)]. \quad (1)$$

Here we used the identity $Q_P = \sum_{k=1}^{T/\delta t}[\delta Q_2(t_k) - \delta Q_{-2}(t_k)]$, and we introduced a variable χ_C, which is conjugated to Q_P and "counts" particle transfers into/out of P. Now, using the Fourier representation of the δ-function, we integrate over $\delta Q_i(t_k)$ and $\chi_i(t_k)$ in (1) and reduce it to a path integral over N and its conjugated χ

$$\langle e^{i\chi_C Q_P}\rangle = \int DN(t) D\chi(t) e^{\int_0^T dt \left(i\chi \dot{N} + N_B H(\chi, N, t)\right)}, \quad (2)$$

where all pre-factors are absorbed into the measure, and

$$H(\chi, N, t) = \left[q_1(t) e_{-\chi} + q_{-2}(t) e_{-(\chi+\chi_C)}\right](1 - N/N_B) \\ + \left[q_{-1} e_{\chi} + q_2 e_{(\chi+\chi_C)}\right] N/N_B. \quad (3)$$

The explicit time-dependence of H is due to the slow periodic changes in $q_1(t)$ and $q_{-2}(t)$.

The exponent in (2) has a factor of N_B in it. Thus for $N_B \to \infty$, the path integral is dominated by the *saddle point* or *classical* values $\chi_{\rm cl}$ and $N_{\rm cl}$,

$$i\dot{\chi}_{\rm cl} = \frac{\partial H(\chi_{\rm cl}, N_{\rm cl}, t)}{\partial N_{\rm cl}}, \quad i\dot{N}_{\rm cl} = -\frac{\partial H(\chi_{\rm cl}, N_{\rm cl}, t)}{\partial \chi_{\rm cl}}. \quad (4)$$

Here the boundary terms disappear, as explained in detail in [13]. Furthermore, since the Hamiltonian (3) is linear in N there are no higher order in $1/N$ corrections.

Since we assume adiabatic variation of q_1 and q_{-2}, the quasi-equilibrium is a good approximation to the exact solution of (4). It corresponds to setting time-derivatives in (4) to zero and treating time as a parameter. For periodic rates variations, this leads to

$$e^{-i\chi_{\rm cl}} \approx \frac{K_- + \sqrt{K^2 + 4q_1 q_2 e_{\chi_C} + 4q_{-1}q_{-2}e_{-\chi_C}}}{2(q_1 + q_{-2}e^{-i\chi_C})}, \quad (5)$$

$$N_{\rm cl} \approx \frac{N_B(q_1 + q_{-2}e^{-i\chi_C})}{q_1 + q_{-2}e^{-i\chi_C} + (q_{-1} + q_2 e^{i\chi_C})e^{2i\chi_{\rm cl}}}, \quad (6)$$

where $\approx$ denotes the accuracy of $O(\omega/q_i)$, $K = q_1 + q_{-2} + q_{-1} + q_2$, $K_- = q_1 + q_{-2} - q_{-1} - q_2$. Since the Hamiltonian is quadratic in its arguments near the saddle point, corrections of the order $O(\omega/q_i)$ in (5, 6) lead to $O[(\omega/q_i)^2]$ contributions to the mgf, setting the accuracy of our results. We now have

$$\langle e^{i\chi_C Q_P}\rangle \approx e^{N_B\left[\int_{\mathbf{c}} \mathbf{A}\cdot d\mathbf{q} + \int_0^T H(\chi_{\rm cl}, N_{\rm cl}, t) dt\right]}, \quad (7)$$

where the vector $\mathbf{A}$, $A_i = i\chi_{\rm cl}(\partial_{q_i} N_{\rm cl})/N_B$, is defined in the space of parameters q_i, and the contour $\mathbf{c}$ is given by $q_i(t)$. For the periodic driving, as we consider here, with a period $T_0 = 2\pi/\omega$ and with fixed q_{-1} and q_2, we rewrite the contour integral as the integral of $F_{q_1,q_{-2}} = \partial_{q_1} A_{-2} - \partial_{q_{-2}} A_1$ over the surface $\mathbf{S_c}$ enclosed by $\mathbf{c}$. Then

$$Z \equiv \langle e^{i\chi_C Q_P}\rangle \approx e^{N_B S_{\rm geom} + N_B S_{\rm cl}}, \quad (8)$$

where

$$S_{\rm geom} = \frac{T}{T_0}\oint_{\mathbf{c}} \mathbf{A}\cdot d\mathbf{q} = \frac{T}{T_0}\int_{\mathbf{S_c}} dq_1 dq_{-2} F_{q_1,q_{-2}}(\mathbf{q}), \quad (9)$$

$$F_{q_1,q_{-2}}(\mathbf{q}) = -\frac{e_{-\chi_C}(e^{i\chi_C}q_2 + q_{-1})}{[4q_1q_2e_{\chi_C} + 4q_{-1}q_{-2}e_{-\chi_C} + K^2]^{3/2}}, \quad (10)$$

$$S_{\rm cl} = \frac{-T}{2T_0}\int_0^{T_0} dt \left\{K - \sqrt{K^2 + 4q_1q_2e_{\chi_C} + 4q_{-1}q_{-2}e_{-\chi_C}}\right\}. \quad (11)$$

The 2-form $F_{q_1,q_{-2}}(\mathbf{q})$ is an analog of the Berry curvature in quantum mechanics. As follows from (9), nonzero Berry curvature is responsible for the reversible component in the particle fluxes. Its presence in our model is due to particle exclusion within the central bin. If k_1 and k_{-2} were independent of N, $F_{q_1,q_{-2}}$ would be zero.

Now moments of the flux between absorbing states can be derived easily by differentiating (8) with respect to χ_C. In particular, the average flux is

$$J = J_{\rm pump} + J_{\rm cl} = N_B\left[\iint_{\mathbf{S_c}} dq_1 dq_{-2}\frac{q_2 + q_{-1}}{T_0 K^3} \right. \\ \left. + \int_0^{T_0} dt\, \frac{q_1q_2 - q_{-1}q_{-2}}{KT_0}\right], \quad (12)$$

where the pump term is due to the particle interactions and the corresponding geometric contribution, while the classical flux would exist even in the stationary limit. Notice that this flux is N_B times its value for a single driven Michaelis-Menten enzyme, studied in [5]. The same holds for the entire mgf, and hence for all flux moments. Thus we refer the reader to [5] for further analysis of the model. Here we note that this scaling is not a coincidence since the current model is, indeed, equivalent to N_B independent enzymes, where N corresponds to the number of enzyme-substrate complexes.

In [5], we used an analogy with the quantum mechanical Berry phase to derive the pump flux (12), while a classical stochastic treatment, which does not require diagonalizing large matrices, is being used now. Existence of these alternative approaches is not surprising because any discrete quantum mechanical system can be mapped onto a mathematically equivalent classical Hamiltonian system [14], and then the Berry phase transforms into a dynamic contribution to the classical action [15]. The contribution of the present derivation is to show that one can derive the classical Hamiltonian for a discrete Markov

chain by considering many identical independent copies of the system and identifying the per-copy contribution by taking the large copy number limit. Alternatively, one can derive the classical reformulation directly from the Schrödinger's equation as well [14, 15].

The reversible ratchet effect. Now we show that the geometric contribution to the mgf is responsible also for the ratchet effect in a periodic potential. Consider a system of noninteracting particles moving in a periodic potential $V(x,t)$, which changes adiabatically with time so that $V(x,t) = V(x,t+T_0)$ and $V(x,t) = V(x+L,t)$. In the overdamped case, the average density of particles satisfies the Fokker-Plank equation

$$\partial_t \rho(x,t) = -\partial_x[A(x,t)\rho(x,t)] + D\partial_x^2 \rho(x,t), \qquad (13)$$

where D is the diffusion coefficient, and $A(x,t) = -\partial_x V(x,t)$ is the force. The current in this model under an adiabatic deformation of the potential was previously studied in [2], and the similarity of the final expression and the Berry phase in quantum mechanics was pointed out. The close connection between the classical ratchets and the Berry phase also has been anticipated in [1, 3]. In our following rederivation, we explicitly show that the ratchet current has its origins in the geometric phase. Namely, it emerges from the complex geometric phase of the particle flux mgf.

To study diffusion without the external field, $A(x,t) = 0$, Refs. [13] derived the path integral for the mgf by discretizing the space into small intervals of length $a \ll L$, indexed by i. Then Poisson transition rates among the neighboring intervals are prescribed in a way that the continuous limit $a \to 0$ recovers the diffusion equation. This reduces the path integral derivation to an already solved problem of stochastic transitions among a discrete set of states. To include the force $A(x,t)$, we assume that it creates an asymmetry in the left and right transition rates. For example, (13) can be recovered if the transition rates are such that during a short time δt the average numbers of particles transfered left and right are $\langle \delta Q_{i\to i-1}\rangle = D\rho(x_i)\delta t/a$ and $\langle \delta Q_{i\to i+1}\rangle = D\rho(x_i)\delta t/a + A(x_i)\rho(x_i)\delta t$, respectively. Then repeating the same steps as in [13] and taking the continuous limit, we find the following path integral representation of the generating function:

$$Z = \langle e^{i\chi_C Q}\rangle = \int D\rho(x,t) D\chi(x,t)\, e^{\int_0^{T_0} dt \int_0^L dx [i\chi\dot{\rho} + H(\rho,\chi)]}, \qquad (14)$$

where Q is the difference between the numbers of particles passing through $x = 0$ in the right and the left directions during the period T_0. The Hamiltonian is

$$H(\rho,\chi) = -iA(x,t)\rho\frac{\partial\chi}{\partial x} + iD\frac{\partial\rho}{\partial x}\frac{\partial\chi}{\partial x} - D\rho\left(\frac{\partial\chi}{\partial x}\right)^2. \qquad (15)$$

The dependence on the counting field χ_C in (15) is hidden in the boundary conditions on χ [12], which, for a periodic system with the spatial period L, are $\rho(L) = \rho(0)$, and $\chi(L) = \chi(0) - \chi_C$. Now, solving the saddle point equations and substituting the result back into the action in the path integral, we write the mgf in a familiar form $Z(\chi_C) = \exp\{S_{\rm geom}(\chi_C) + S_{\rm cl}(\chi_C)\}$, where $S_{\rm geom}(\chi_C) = \int_0^{T_0} dt \int_0^L dx\, (i\chi_{\rm cl}\dot{\rho}_{\rm cl})$, and $S_{\rm cl}(\chi_C) = \int_0^{T_0} dt \int_0^L dx\, H(\rho_{\rm cl}(\chi_C), \chi_{\rm cl}(\chi_C))$.

The analysis simplifies if we are interested only in mean currents, rather than in their fluctuations. Then we consider $\chi_C \ll 1$ and find the contribution to $\log Z$ that is linear in it. In fact, only $S_{\rm geom}$ has this contribution in our case. To determine it, it is sufficient to find $\rho_{\rm cl}(x,t)$ to the zeroth order and $\chi_{\rm cl}(x,t)$ to the first order in χ_C. This results in $\rho_{\rm cl}(x,t) \approx [Q_0/R_-(t)]e^{-V(x,t)/k_BT}$, $\chi_{\rm cl}(x,t) \approx [-\chi_C/R_+(t)]\int_0^x e^{V(x',t)/k_BT}dx'$, where $R_\pm(t) = \int_0^L e^{\pm V(x,t)/k_BT}dx$, and $Q_0 = \int_0^L \rho_{\rm cl}(x,t)|_{\chi_C=0}dx$ is the number of particles per unit cell. This leads to $Z(\chi_C, T_0) = \exp[i\chi_C J T_0 + O(\chi_C^2)]$, where the terms in $O(\chi_C^2)$ can reveal the higher order cumulants, and the average current $J = -(i/T_0)(\partial_{\chi_C}\log Z)_{\chi_C=0}$ is

$$J = \frac{Q_0}{2T_0}\int_0^{T_0} dt \int_0^L dx\,(\partial_t v\,\partial_x u - \partial_x v\,\partial_t u), \qquad (16)$$

where we introduced $u(x,t) = \frac{1}{R_-(t)}\int_0^x e^{-V(x',t)/k_BT}dx'$ and $v(x,t) = \frac{1}{R_+(t)}\int_0^x e^{V(x',t)/k_BT}dx'$ and used the property $u(L,t) = v(L,t) = 1$. The integrand in (16) is a pure curl of a vector $\mathbf{A}$ with components $A_x = v(x,t)\partial_x u(x,t)$ and $A_t = v(x,t)\partial_t u(x,t)$ defined in the two dimensional space-time. Thus the current can be expressed as

$$J = \frac{Q_0}{2T_0}\oint_{\mathbf{c}} \mathbf{A}\cdot\mathbf{dr}, \qquad (17)$$

where $\mathbf{dr} = (dx, dt)$, and $\mathbf{c}$ is the contour that encloses a space-time cell with boundaries at $x = 0, L$ and $t = 0, T_0$.

For a uniformly shifting potential $V(x,t) = V(x - tL/T_0)$, $R_\pm$ are time-independent, and the integration in (16) leads to $J = Q_0/T_0 - (Q_0/T_0)L^2/(R_+R_-)$. The first term in this expression is the quantized contribution which is dominating in the limit of a large potential amplitude. In [3], this quantization of the classical ratchet current was connected to the Chern number of the Bloch band related to the potential $V(x)$.

Pump current in the SIS epidemiological model. In a final calculation, we show how the stochastic path integral allows derivation of pump-like effects in novel scenarios; specifically where, unlike in our first example, the system cannot be factored into non-interacting identical stochastic subsystems. We consider the standard Susceptible-Infected-Susceptible (SIS) mechanism of an infection outbreak, which is a good model for influenza. State of the art epidemiological modeling uses deterministic dynamics [16], which tracks only fractions of populations in various states during an outbreak progression.

However, it is understood that stochasticity may be essential. Thus here we discuss if stochasticity, and especially effects due to slow variability of the infectivity and the recovery rates, can affect disease outbreaks.

Let's denote infected individuals by I and their number by N. The disease spreads due to a permanent infection source and because it can be transmitted by the infected individuals. All infected people eventually recover.

Thus the full kinetic scheme is

1. $\emptyset \to I$; rate k_1 (permanent infection source);
2. $I \to \emptyset$; rate per infected individual k_2 (recovery);
3. $I \to I+I$; rate per infected individual k_3 (infection spread by contacts).

Here k_i are independent of N because we assume that outbreaks are small in comparison to the total population size (still $N \gg 1$ is possible). This requires that $k_2 > k_3$, so that, if stochasticity is unimportant, the deterministic steady state solution is $N_{\rm st} = k_1/(k_2 - k_3)$, and the stationary flux into and out of the infected state is $J_{\rm st} = k_2 N_{\rm st} = k_1 k_2/(k_2 - k_3)$. This model is a birth-death process, and, with time-independent rates, it has been studied extensively. Here we are interested in estimating (possibly substantial) effects of rate time-dependence. The Hamiltonian in the path integral for this model is

$$H(\chi, N, t) = k_1(t)e_{-\chi} + k_2 N e_{(\chi+\chi_C)} + k_3(t) N e_{-\chi}, \quad (18)$$

where χ is the conjugated variable to N and χ_C counts the flux out of I. With $N \gg 1$, we can use the saddle point analysis, which is exact since H is linear in N.

Now consider a periodic time dependence of the rates k_i, which may be due to the time-of-day or seasonal effects. For simplicity, we assume that only k_1 and k_3 vary, and the recovery rate k_2 remains constant. As before, the mgf has both the classical and the geometric contributions, i.e. $Z = \exp[S_{\rm cl} + S_{\rm geom}]$. The classical one is the average of the stationary mgf over the period of the rates variation, T_0, while the geometric one is again an integral over the surface $\mathbf{S_c}$ inside the contour enclosed by $k_1(t)$ and $k_3(t)$:

$$S_{\rm cl} + S_{\rm geom} = \frac{T}{T_0}\int_0^{T_0} dt H[\chi_{\rm cl}(t), N_{\rm cl}(t), t] + \frac{T}{T_0}\iint_{\mathbf{S_c}} dk_1 dk_3 F_{k_1,k_3}(\mathbf{k}), \quad (19)$$

$$F_{k_1,k_3}(\mathbf{k}) = \frac{k_2(K_- - 2k_3 e_{\chi_C} - \kappa)}{2k_3^2\kappa^2}, \quad (20)$$

$$H(\chi_{\rm cl}, N_{\rm cl}, t) = \frac{k_1(K_- - \kappa)}{2k_3}, \quad (21)$$

where now $K_- = k_2 - k_3$, and $\kappa = \sqrt{K_-^2 - 4k_2k_3e_{\chi_C}}$. This corresponds to the mean flux $J = J_{\rm pump} + J_{\rm cl}$, where $J_{\rm pump} = \frac{1}{T_0}\iint_{\mathbf{S_c}} \frac{dk_1 dk_3\, k_2}{K_-^3}$ is the pump flux due to the geometric contribution, and the classical flux is $J_{\rm cl} = 1/T_0 \int_0^T J_{\rm st} dt$. Notice, in particular, that $J_{\rm pump} \propto K_-^{-3}$, and it can become very large near $K_- = 0$. Fluctuations are easy to compute as well by differentiating (19).

Conclusion. Based on the stochastic path integral technique, we built the theory of geometric fluxes in classical stochastic kinetics, and we proposed a general approach for identification and calculation of pump-like currents, including the familiar reversible ratchet, as well as new phenomena. In the adiabatic limit, the full counting statistics of pump fluxes is provided by the term that depends on the choice of the contour in the parameter space, but does not depend on the rate of the motion along this contour, and thus has a geometric origin. The stationary saddle point approximation of the path integral is sufficient for calculations of this geometric contribution in the case of a large number of particles. This approach leads to the complete theory of reversible effects in nonequilibrium statistical physics. It will open doors to a study of such poorly understood systems as ratchets with interacting diffusing particles, or epidemiological models on complex social networks with time-dependent parameters.

We thank M. Wall, F. Alexander, N. Hengartner and R. D. Astumian for useful discussions and critical reading of this text. This work was funded in part by DOE under Contract No. DE-AC52-06NA25396. IN was further supported by NSF Grant No. ECS-0425850.

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